ma
the
ma
tisch

cen
trum

amsterdam 1979

AFDELING ZUIVERE WISKUNDE ZN 88/79 MARCH
(DEPARTMENT OF PURE MATHEMATICS)

J. DE VRIES

A NOTE ON WEIGHT FUNCTIONS ON LOCALLY COMPACT GROUPS

stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZN 88/79 MARCH
(DEPARTMENT OF PURE MATHEMATICS)

J. DE VRIES

A NOTE ON WEIGHT FUNCTIONS ON LOCALLY COMPACT GROUPS

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 43A99, 54H15

A note on weight functions on locally compact groups

by

J. de Vries

ABSTRACT

In their construction of universal linear transformation groups, BAAYEN and DE GROOT use a certain type of square integrable functions on locally compact groups. Though it was known that a locally compact group G admits such a function if and only if G is sigma-compact, the characterization of groups admitting continuous functions of this type was still an open problem. In this note, we solve this problem by showing that every function of this type on any locally compact, sigma-compact group can be transformed into a continuous one.

KEY WORDS & PHRASES: *locally compact sigma-compact Hausdorff topological group, square-integrable function, weight function, universal linear transformation group.*

A NOTE ON WEIGHT FUNCTIONS ON LOCALLY COMPACT GROUPS

In this note, G denotes a locally compact, sigma-compact Hausdorff topological group. Fix a right Haar measure μ on G, and let $L^2(G)$ be the space of (equivalence classes of) μ-square integrable functions.

A *weight function* on G is an element w in $L^2(G)$ having the following properties

(i) $\forall t \in G$: $w(t) > 0$;

(ii) $\forall s,t \in G$: $w(st) \geq w(s)w(t)$;

(iii) the function $t \mapsto w(t)^{-1}: G \to \mathbb{R}^+$ is locally bounded (i.e. is bounded on compact subsets of G).

The concept of a weight function, or rather, a function $w \in L^2(G)$ satisfying the conditions (i) and (ii) was introduced in [1] by P.C. BAAYEN and J. DE GROOT in order to prove the following result: for every cardinal number κ there exists a Hilbert space H_κ such that, if G acts as a transformation group on a metrizable space M of (topological) weight $\leq \kappa$, then M can be topologically embedded in H_κ in such a way that the transformations $\gamma \in G$ become restrictions of invertible bounded linear operators on H_κ. Condition (iii) was used in [4] in order to show that the bounded linear operators on H_κ referred to above can be obtained from a continuous linear action of G on H_κ. In [3] it was shown that on every locally compact, sigma-compact group an element $w \in L^2(G)$ exists, satisfying the conditions (i) and (ii). (Sigma-compactness of G is, in fact, a necessary condition for the existence of $w \in L^2(G)$ satisfying (i).) Implicit in the construction in [3] is also that an element $w \in L^2(G)$ exists satisfying the conditions (i), (ii) and (iii); cf. [4]. *Thus, a locally compact group admits a weight function iff the group is sigma-compact.*

In [1] the problem has been raised whether there exists a *continuous* weight function on G (observe that condition (iii) is automatically fulfilled for every continuous function satisfying condition (i)). Every locally compact abelian group which is either separable or compactly generated admits a continuous weight function; cf. the proof of Theorem 3.3 in [1]. In this note, we prove:

THEOREM. *If* G *is a locally compact, sigma-compact Hausdorff topological group, then there exists a continuous weight function on* G.

PROOF. According to the results quoted above there exists a weight function w on G. By right invariance of the Haar measure μ we have for every $t \in G$:

$$\|w\|_2^2 = \int_G w(st)^2 d\mu(s) \geq w(t)^2 \int_G w(s)^2 d\mu(s) = w(t)^2 \|w\|_2^2.$$

Since $\|w\|_2 \neq 0$ it follows that $w(t) \leq 1$ for every $t \in G$. Consequently, if we put

$$w_1(t) := \left(w(t)w(t^{-1})\right)^2$$

for $t \in G$, then w_1 is a Borel function such that $0 < w_1(t)^p \leq w_1(t) \leq w(t)^2$ for every $t \in G$ and $p \geq 1$. It follows that w_1^p is integrable for every $p \geq 1$. Moreover, it is easily checked that w_1 is a weight function and that w_1 is symmetric, i.e. $w_1(t^{-1}) = w_1(t)$ for all $t \in G$.
Next, put $w_2 := cw_1 * w_1^2$, that is (cf. [2; 20.32 (c)]):

$$w_2(t) = c \int_G w_1(ts^{-1}) w_1(s)^2 d\mu(s),$$

where c is a positive constant which will be determined below. Since w_1 is symmetric and $w_1, w_1^2 \in L^2(G)$, [2; 20.32 (e)] implies that w_2 is well-defined and that, in fact, $w_2 \in C_0(G)$. We shall complete the proof by showing that w_2 is a weight function. To this end, observe that conditions (i) and (ii) for w_1 imply that for all $t \in G$

$$w_2(t) \geq c \int_G w_1(t) w_1(s^{-1}) w_1(s)^2 d\mu(s) = cw_1(t) \int_G w_1(s)^3 d\mu(s)$$

and

$$w_2(t) \leq c \int_G w_1(t) w_1(s) d\mu(s) = cw_1(t) \int_G w_1(s) d\mu(s).$$

Putting

$$a := \int_G w_1(s)^3 d\mu(s), \quad b := \int_G w_1(s) d\mu(s)$$

we have for all $t \in G$

$$(*) \qquad ac\, w_1(t) \le w_2(t) \le bc\, w_1(t),$$

where a and b are positive constants (finite, of course).

Clearly, $(*)$ implies that w_2 satisfies conditions (i) and (iii) in the definition of a weight function, since w_1 does. Moreover, for all $s,t \in G$ we have

$$w_2(st) \ge ac\, w_1(st) \ge ac\, w_1(s)w_1(t) \ge \frac{a}{b^2c} w_2(s)w_2(t).$$

Hence, if we set $c := ab^{-2}$, then w_2 satisfies condition (iii). Finally, as w_2 is dominated by a multiple of w_1 and w_1 is in $L^2(G)$, also $w_2 \in L^2(G)$. □

<u>REMARK</u>. Applying the first part of the proof to w_2, we see that G admits a weight function w_3 such that w_3 is symmetric and $w_3 \in C_0(G) \cap \bigcap_{p\ge 1} L^p(G)$.

REFERENCES

[1] BAAYEN, P.C. & J. DE GROOT, Linearization of locally compact transformation groups in Hilbert space, *Math. Systems Theory* <u>2</u> (1968), 363-379.

[2] HEWITT, E & K.A. ROSS, *Abstract Harmonic Analysis I*, Springer-Verlag, Berlin-Heidelberg-New York, 1963.

[3] PAALMAN-DE MIRANDA, A.B., A note on W-groups, *Math. Systems Theory* <u>5</u> (1971), 168-171.

[4] DE VRIES, J., A note on topological linearization of locally compact transformation groups in Hilbert space, *Math. Systems Theory* <u>6</u> (1972), 49-59.